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Statistical Control Paradigm for Aerospace Structures Under Impulsive Disturbances

Khanh D. Pham[*] and Lawrence M. Robertson[†]

Air Force Research Laboratory, Kirtland AFB, New Mexico, 87117, U.S.A.

In this paper the newly developed statistical control theory is revisited to autonomously control the satellite attitude as well as to provide a means of actively attenuating impulsive disturbances caused by servicing dock and space debris. Simulations are performed using several docking and collision scenarios. The simulation results indicate that the existing attitude control system with an innovative and robust statistical controller design shows significant promise for use in attitude hold mode operation despite the presence of impulsive disturbances.

Nomenclature

H_C Angular momentum of the satellite about its center of mass measured in inertial frame $\{N\}$
H_W Angular momentum of the wheel cluster
J_C Satellite inertia
J_W Reaction wheel inertia
$\omega^{S/N}$ Angular velocity vector in fixed-body reference frame $\{S\}$
ω^W Angular velocity of the reaction wheels
L Wheel orientation matrix
τ_{thrt} Absolute torque due to the thrusters
τ_W Absolute torque due to the reaction wheels
τ_{grav} Torque due to the earth gravity gradient
τ_{mag} Torque due to the earth magnetic field
τ_{aero} Torque due to atmospheric drag
τ_{srp} Torque due to solar radiation pressure

I. Introduction

Recent work[1–5] reported at several American Control Conferences showed that the statistical control design has been performing quite competitively with other modern control techniques. In the statistical control formulation, a state-feedback controller is designed to minimize the objective function representing a linear combination of finite cumulant indices of a finite horizon integral quadratic performance measure associated to a linear stochastic system. A dynamic programming approach is used to obtain the optimal control solution. This control algorithm is then applied to attenuate dynamical effects due to impulsive disturbances. Motivations for this arise from the need to study control problems related to antennas on the space station subject to impact from space debris and active damping of vibrations of flexible structures caused by impact forces. The outline of the paper goes as follows. First, a brief modeling of satellite with reaction wheels and thrusters will be given below. Then some stabilizing controllers minimizing the first three cost cumulant indices from space disturbances to the controlled output are subsequently designed. Finally, the results of simulation are presented along with some discussions.

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[*]Aerospace Engineer, Space Vehicles Directorate, 3550 Aberdeen Ave SE, and AIAA Member.
[†]Dynamics & Controls Program Manager, Space Vehicles Directorate, 3550 Aberdeen Ave SE, and AIAA Senior Member.

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II.   Mathematical Model of Satellite with Reaction Wheels and Thrusters

This section gives the reader an overview of a simple mathematical model for center of mass steering and attitude control of a Low-Earth-Orbiting (LEO) satellite whose detailed definition is given in Won.[6] This model assumes that the largest disturbance is the gravity gradient torque while other space disturbances are small and can thus be modeled as a stationary Wiener random process. Moreover the internal torques generated by a cluster of four reaction wheels and three thrusters are for use in controlling the attitude of the satellite. Then the angular momentum of the satellite with three thrusters and a cluster of four reaction wheels becomes

$$\vec{H}_C = J_C \vec{\omega}^{S/N} + L^T J_W \vec{\omega}^W \tag{1}$$

where J_W is the mass moments of inertia of the reaction wheels; $\vec{\omega}^W$ is the angular velocity of the wheels; and L is the wheel orientation matrix

$$J_W = \begin{bmatrix} J_{11}^W & 0 & 0 & 0 \\ 0 & J_{22}^W & 0 & 0 \\ 0 & 0 & J_{33}^W & 0 \\ 0 & 0 & 0 & J_{44}^W \end{bmatrix}, \qquad L = \begin{bmatrix} \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \\ -\sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \\ -\cos\alpha\sin\beta & -\sin\alpha\sin\beta & \cos\beta \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix}. \tag{2}$$

Furthermore, the transpose of the wheel orientation matrix L is denoted by

$$L^T = \begin{bmatrix} L_t(1,1) & L_t(1,2) & L_t(1,3) & L_t(1,4) \\ L_t(2,1) & L_t(2,2) & L_t(2,3) & L_t(2,4) \\ L_t(3,1) & L_t(3,2) & L_t(3,3) & L_t(3,4) \end{bmatrix}$$

with $\alpha = 45$ degrees and $\beta = 54.74$ degrees. By taking the inertial time derivative of (1), the nonlinear satellite attitude dynamics can be described in terms of vector and matrix notations

$$J_C \dot{\omega}^{S/N} + L^T J_W \dot{\omega}^W + \tilde{\omega}^{S/N}\left(J_C \omega^{S/N} + L^T J_W \omega^W \right) = \tau_{\text{thrt}} + \tau_{\text{grav}} + w\,, \tag{3}$$

where the stationary Wiener process w and the cross product matrix $\tilde{\omega}^{S/N}$ are defined as

$$w \triangleq \tau_{\text{aero}} + \tau_{\text{mag}} + \tau_{\text{srp}}\,, \qquad \tilde{\omega}^{S/N} \triangleq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Furthermore, the angular momentum of the wheel cluster H_W and the absolute torque due to the reaction wheels τ_W are related by

$$\dot{H}_W = \tau_W\,, \tag{4}$$

$$H_W = J_W \omega^W + J_W L \omega^{S/N}\,. \tag{5}$$

From the equation (5), it is possible to write

$$\dot{\omega}^W = J_W^{-1} \tau_W - L\dot{\omega}^{S/N}\,, \tag{6}$$

which results in

$$\left(J_C - L^T J_W L \right) \dot{\omega}^{S/N} = -\tilde{\omega}^{S/N}\left(J_C \omega^{S/N} + L^T J_W \omega^W \right) - L^T \tau_W + \tau_{\text{thrt}} + \tau_{\text{grav}} + w\,. \tag{7}$$

Let $J_\triangle \triangleq J_C - L^T J_W L$. The equations (6) and (7) can then be rewritten as follows

$$\dot{\omega}^{S/N} = -J_\triangle^{-1}\tilde{\omega}^{S/N}\left(J_C \omega^{S/N} + L^T J_W \omega^W \right) - J_\triangle^{-1} L^T \tau_W + J_\triangle^{-1}\tau_{\text{thrt}} + J_\triangle^{-1}\tau_{\text{grav}} + J_\triangle^{-1} w\,, \tag{8}$$

$$\dot{\omega}^W = J_W^{-1}\tau_W + L J_\triangle^{-1}\tilde{\omega}^{S/N}\left(J_C \omega^{S/N} + L^T J_W \omega^W \right) + L J_\triangle^{-1} L^T \tau_W - L J_\triangle^{-1}\tau_{\text{thrt}} - L J_\triangle^{-1}\tau_{\text{grav}} - L J_\triangle^{-1} w\,. \tag{9}$$

Moreover, the orientation of the body-fixed reference frame S relative to the orbital reference frame O can also be described by introducing the time dependence of quaternions $q = (q_1, q_2, q_3, q_4)^T$. Note that the orbital reference frame $O = \{O_x, O_y, O_z\}$ is rotating about the O_y axis with respect to the Newtonian inertial reference frame N at the orbital rate ω_0. The axes of O are chosen such that the roll axis O_x is in the flight direction, the pitch axis O_y is perpendicular to the orbital plane in the negative direction and the yaw axis O_z points from the satellite to the earth center. The benefits of using quaternions as opposed to Euler angles include no inherent geometric singularity and suitability of onboard real-time computation. It is shown from Wie[10] that the kinematic differential equations for quaternions can be shown as follows

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} .
\tag{10}
$$

where $\omega^{S/O} = (\omega_x, \omega_y, \omega_z)^T$ is the rotation rate of the body-fixed reference frame S with respect to the orbit reference frame O, expressed in the body-fixed reference frame. Moreover, the direction cosine matrix, $C^{S/O}$ is needed for a coordinate transformation that maps the angular velocity from $\omega^{S/O}$ to $\omega^{S/N}$

$$
\omega^{S/N} = \omega^{S/O} + C^{S/O} \omega^{O/N} ,
\tag{11}
$$

$$
C^{S/O} = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & q_2^2 - q_1^2 - q_3^2 + q_4^2 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & q_3^2 - q_1^2 - q_2^2 + q_4^2 \end{bmatrix} .
\tag{12}
$$

Thus, when

$$
\omega^{O/N} = \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix} ,
\tag{13}
$$

is the inertial rotation rate of the orbital reference frame with respect to the inertial frame, the ω_x, ω_y, and ω_z in (10) can be replaced according to the following equations

$$
\omega_x = \omega_1 + 2\omega_0(q_1 q_2 + q_3 q_4) ,
\tag{14}
$$

$$
\omega_y = \omega_2 + \omega_0(q_2^2 - q_1^2 - q_3^2 + q_4^2) ,
\tag{15}
$$

$$
\omega_z = \omega_3 + 2\omega_0(q_2 q_3 - q_1 q_4) .
\tag{16}
$$

The gravity gradient torque about the satellite's mass center is given by Wie[10]

$$
\tau_{\text{gravity}} = 3\omega_0^2 \vec{O}_z \times J_C \vec{O}_z ,
\tag{17}
$$

where the vector O_z, represents the projection of the orbit radius vector onto the body-fixed reference frame, S

$$
\vec{O}_z = C_{13}^{S/O} \hat{s}_1 + C_{23}^{S/O} \hat{s}_2 + C_{33}^{S/O} \hat{s}_3 .
\tag{18}
$$

Thus, the equation (17) becomes

$$
\begin{aligned}
\tau_{\text{gravity}} &= 3\omega_0^2 \begin{bmatrix} 0 & -C_{33}^{S/O} & C_{23}^{S/O} \\ C_{33}^{S/O} & 0 & -C_{13}^{S/O} \\ -C_{23}^{S/O} & C_{13}^{S/O} & 0 \end{bmatrix} \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} \begin{bmatrix} C_{13}^{S/O} \\ C_{23}^{S/O} \\ C_{33}^{S/O} \end{bmatrix} , \\
&= 3\omega_0^2 \begin{bmatrix} 0 & -(q_3^2 - q_1^2 - q_2^2 + q_4^2) & 2(q_2 q_3 + q_1 q_4) \\ (q_3^2 - q_1^2 - q_2^2 + q_4^2) & 0 & -2(q_1 q_3 - q_2 q_4) \\ -2(q_2 q_3 + q_1 q_4) & 2(q_1 q_3 - q_2 q_4) & 0 \end{bmatrix} \\
&\qquad\qquad \cdot \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} \begin{bmatrix} 2(q_1 q_3 - q_2 q_4) \\ 2(q_2 q_3 + q_1 q_4) \\ (q_3^2 - q_1^2 - q_2^2 + q_4^2) \end{bmatrix} .
\end{aligned}
\tag{19}
$$

Finally the the rotational motions of the satellite with three rotational degrees of freedom can be described by the kinematic differential equations

$$\dot{q}_1 = \frac{1}{2}\left(\omega_1 q_4 - \omega_2 q_3 + \omega_3 q_2 + \omega_0 q_3\right), \tag{20}$$

$$\dot{q}_2 = \frac{1}{2}\left(\omega_1 q_3 + \omega_2 q_4 - \omega_3 q_1 + \omega_0 q_4\right), \tag{21}$$

$$\dot{q}_3 = \frac{1}{2}\left(-\omega_1 q_2 + \omega_2 q_1 + \omega_3 q_4 - \omega_0 q_1\right), \tag{22}$$

$$\dot{q}_4 = \frac{1}{2}\left(-\omega_1 q_1 - \omega_2 q_2 - \omega_3 q_3 - \omega_0 q_2\right), \tag{23}$$

$$J_\triangle \dot{\omega}^{S/N} = -\tilde{\omega}^{S/N}\left(J_C \omega^{S/N} + L^T J_W \omega^W\right) - L^T \tau_W + \tau_{\mathrm{thrt}} + \tau_{\mathrm{grav}} + w, \tag{24}$$

$$\dot{\omega}^W = LJ_\triangle^{-1}\tilde{\omega}^{S/N}\left(J_C \omega^{S/N} + L^T J_W \omega^W\right) + \left(LJ_\triangle^{-1}L^T + J_W^{-1}\right)\tau_W - LJ_\triangle^{-1}\tau_{\mathrm{thrt}} - LJ_\triangle^{-1}\tau_{\mathrm{grav}} - LJ_\triangle^{-1}w. \tag{25}$$

which can be further rewritten compactly using the mapping $f : \mathbb{R}^{11} \times \mathbb{R}^7 \times \mathbb{R} \mapsto \mathbb{R}^{11}$ whose rule of action is given by

$$\dot{x} = f(x, u, t) + v, \qquad t \in [t_0, t_f] \tag{26}$$

wherein the state variable, $x \in \mathbb{R}^{11}$; control variable constrained to some compact set, $u \in \mathbb{R}^7$; and combined torque disturbances, $v \in \mathbb{R}^3$ are defined by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{bmatrix} \triangleq \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_1^W \\ \omega_2^W \\ \omega_3^W \\ \omega_4^W \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} \triangleq \begin{bmatrix} \tau_W(1) \\ \tau_W(2) \\ \tau_W(3) \\ \tau_W(4) \\ \tau_{\mathrm{thrt}}(1) \\ \tau_{\mathrm{thrt}}(2) \\ \tau_{\mathrm{thrt}}(3) \end{bmatrix}, \qquad v = Gw \triangleq \begin{bmatrix} 0_{4\times3} \\ J_\triangle^{-1} \\ -LJ_\triangle^{-1} \end{bmatrix} w,$$

and t_0 and t_f are the beginning and ending mission lifetimes. Since the high fidelity nonlinear model in (20)-(25) is too complicated for use in control design. It is necessary to develop a simpler linearized model which can approximate the system over a wide range of conditions. The following linearized model assumes a circular orbit, and linearization is performed about the equilibrium nadir-pointing attitude. Clearly, it is easy to see that there exist two trivial equilibrium points (x^*, u^*) for the deterministic system $\dot{x} = f(x, u, t)$

$$x^* = (0, 0, 0, 1, 0, -\omega_0, 0, 0, 0, 0, 0), \qquad\qquad x^* = (0, 0, 0, -1, 0, -\omega_0, 0, 0, 0, 0, 0),$$
$$u^* = (0, 0, 0, 0, 0, 0, 0), \qquad\qquad u^* = (0, 0, 0, 0, 0, 0, 0),$$

such that

$$f(x^*, u^*, t) = 0, \qquad t \in [t_0, t_f].$$

Once the satellite is in orbit, the satellite is controlled to keep its pointing direction about a pre-computed orientation. In this case, a reference orientation is defined as a state x^* in which the deviation of orientation is preferably zero under a steady state condition. The control objective is to regulate the orientation of the satellite about these desired orientations. The satellite's tracking of itself to the prescribed orientation is equivalent to the expression $x - x^*$ being zero. This indicates that linearization of (26) about some reference orientation can be useful for studying the effect of perturbation away from the desired orientation

and remains accurate for sizable changes in the satellite's angular velocity. Reaction wheels and electric thrusters can be used to actively drive observed perturbations back to zero. The expansion of the right-hand side of the equation (26) in a first-order Taylor series expansion in terms of small perturbations proceeds with

$$\delta x = x - x^*,$$
$$\delta u = u - u^*,$$

where the reference orientation and the nominal control input are

$$
x^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \\ x_6^* \\ x_7^* \\ x_8^* \\ x_9^* \\ x_{10}^* \\ x_{11}^* \end{bmatrix}, \qquad
u^* = \begin{bmatrix} u_1^* \\ u_2^* \\ u_3^* \\ u_4^* \\ u_5^* \\ u_6^* \\ u_7^* \end{bmatrix}.
$$

The linearization of (20)-(25) about the reference orientation yields

$$\delta \dot{x}(t) = A(x^*(t), u^*(t))\delta x(t) + B(x^*(t), u^*(t))\delta u(t) + v(t), \qquad \delta x(t_0) = x_0, \qquad (27)$$

where matrix coefficients $A(\cdot, \cdot)$, and $B(\cdot, \cdot)$ are given by

$$A(x^*(t), u^*(t)) = \left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)} = \begin{bmatrix} A_{x_{1:4}x_{1:4}} & A_{x_{1:4}x_{5:7}} & A_{x_{1:4}x_{8:11}} \\ A_{x_{5:7}x_{1:4}} & A_{x_{5:7}x_{5:7}} & A_{x_{5:7}x_{8:11}} \\ A_{x_{8:11}x_{1:4}} & A_{x_{8:11}x_{5:7}} & A_{x_{8:11}x_{8:11}} \end{bmatrix}, \qquad (28)$$

$$B(x^*(t), u^*(t)) = \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} = \begin{bmatrix} B_{x_{1:4}u_{1:4}} & B_{x_{1:4}u_{5:7}} \\ B_{x_{5:7}u_{1:4}} & B_{x_{5:7}u_{5:7}} \\ B_{x_{8:11}u_{1:4}} & B_{x_{8:11}u_{5:7}} \end{bmatrix}. \qquad (29)$$

where

$$A_{x_{1:4}x_{1:4}} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & 0 \\ -\omega_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_{x_{1:4}x_{5:7}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_{x_{1:4}x_{8:11}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{x_{5:7}x_{1:4}} = (J_C - L^T J_W L)^{-1} \begin{bmatrix} 6\omega_0^2(J_{33} - J_{22}) & 0 & 0 & 0 \\ 0 & 6\omega_0^2(J_{33} - J_{11}) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{x_{5:7}x_{5:7}} = (J_C - L^T J_W L)^{-1} \begin{bmatrix} 0 & 0 & -\omega_0(J_{22} - J_{33}) \\ 0 & 0 & 0 \\ -\omega_0(J_{11} - J_{22}) & 0 & 0 \end{bmatrix},$$

$$A_{x_{5:7}x_{8:11}} = (J_C - L^T J_W L)^{-1} \begin{bmatrix} \omega_0 L_t(3,1)J_{11}^W & \omega_0 L_t(3,2)J_{22}^W & \omega_0 L_t(3,3)J_{33}^W & \omega_0 L_t(3,4)J_{44}^W \\ 0 & 0 & 0 & 0 \\ -\omega_0 L_t(1,1)J_{11}^W & -\omega_0 L_t(1,2)J_{22}^W & -\omega_0 L_t(1,3)J_{33}^W & -\omega_0 L_t(1,4)J_{44}^W \end{bmatrix},$$

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$$A_{x_{8:11}x_{1:4}} = -LA_{x_{5:7}x_{1:4}}, \qquad A_{x_{8:11}x_{5:7}} = -LA_{x_{5:7}x_{5:7}}, \qquad A_{x_{8:11}x_{8:11}} = -LA_{x_{5:7}x_{8:11}},$$

$$B_{x_{1:4}u_{1:4}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_{x_{1:4}u_{5:7}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_{x_{5:7}u_{1:4}} = -(J_C - L^T J_W L)^{-1} L^T, \qquad B_{x_{5:7}u_{5:7}} = (J_C - L^T J_W L)^{-1},$$

$$B_{x_{8:11}u_{1:4}} = L(J_C - L^T J_W L)^{-1} L^T + J_W^{-1}, \qquad B_{x_{8:11}u_{5:7}} = -L(J_C - L^T J_W L)^{-1}.$$

III. Model of Satellite In Events of Servicing Dock and Debris Impacts

Next, the conservation of angular momentum is utilized in the event of docking and space debris impacts. The assumptions made herein include not limited to the case where a servicing satellite that is slowly attaching to the satellite, or the case where high speed debris is embedded in a perfectly inelastic collision. Thus, the angular momentum immediately before and after the impact event must match

$$\left[\vec{H}_{Sat} + \vec{H}_{Docking/Debris} \right]_{Pre-Impact} = \left[\vec{H}_{Sat} + \vec{H}_{Docking/Debris} \right]_{Post-Impact}. \tag{30}$$

It is well to note that the angular momentum of the satellite before and after the impact are

$$\vec{H}_{Sat}|_{Pre-Impact} = 0,$$
$$\vec{H}_{Sat}|_{Post-Impact} = J_{Sat+Docking/Debris}\vec{\omega}^{S/N},$$

and the angular momentum of either a servicing satellite or space debris is given by

$$\vec{H}_{Docking/Debris}|_{Pre-Impact} = \vec{r}_{Docking/Debris} \times m_{Docking/Debris}\vec{v}_{Docking/Debris},$$
$$\vec{H}_{Docking/Debris}|_{Post-Impact} = 0,$$

wherein $\vec{r}_{Docking/Debris}$ is the position vector from the satellite's center of mass to the point of impact and $m_{Docking/Debris}\vec{v}_{Docking/Debris}$ is the linear momentum of either a servicing satellite or space debris. Therefore, the equality (30) can be readily reduced to

$$\vec{r}_{Docking/Debris} \times m_{Docking/Debris}\vec{v}_{Docking/Debris} = J_{Sat+Docking/Debris}\vec{\omega}^{S/N}, \tag{31}$$

provided $J_{Sat+Docking/Debris} = J_C + \triangle J_C$ is the combined moment of inertia of the satellite, servicing satellite or space debris. Resolving $\vec{r}_{Docking/Debris}$ and $\vec{v}_{Docking/Debris}$ into the body-fixed reference frame S

$$\vec{r}_{Docking/Debris} = r_1\hat{s}_1 + r_2\hat{s}_2 + r_3\hat{s}_3,$$
$$\vec{v}_{Docking/Debris} = v_1\hat{s}_1 + v_2\hat{s}_2 + v_3\hat{s}_3,$$

yields the angular velocities of the satellite in the event of either docking or collision

$$\omega_1 = m_{Docking/Debris}\frac{r_2v_3 - r_3v_2}{J_{Sat+Docking/Debris}^{11}}, \tag{32}$$

$$\omega_2 = m_{Docking/Debris}\frac{r_1v_3 - r_3v_1}{J_{Sat+Docking/Debris}^{22}}, \tag{33}$$

$$\omega_3 = m_{Docking/Debris}\frac{r_1v_2 - r_2v_1}{J_{Sat+Docking/Debris}^{33}}. \tag{34}$$

From the equation (31), it is observed that a soft docking would be possible if the servicing satellite had small mass, docked slowly and was close to the center of mass of the satellite. For simplicity, impact forces

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caused by either the docking event or space debris are modeled as finite duration events and it is further fruitful to assume that each impact duration is very short and is denoted as $t_{i+1} - t_i \triangleq h_i$. Thus, the impact can be treated as an impulsive event and the impact force $w_d(t_i)\delta(t - t_i)$ generated by the ith impact and the resulting angular velocity of the satellite $\omega_1(t_i)$, $\omega_2(t_i)$, and $\omega_3(t_i)$ caused by the ith impact force is defined as

$$\omega_1(t_i) = m_{Docking/Debris}(t_i)\frac{r_2(t_i)v_3(t_i) - r_3(t_i)v_2(t_i)}{J_{Sat+Servicing/Debris}^{11}(t_i)},$$

$$\omega_2(t_i) = m_{Docking/Debris}(t_i)\frac{r_1(t_i)v_3(t_i) - r_3(t_i)v_1(t_i)}{J_{Sat+Servicing/Debris}^{22}(t_i)},$$

$$\omega_3(t_i) = m_{Docking/Debris}(t_i)\frac{r_1(t_i)v_2(t_i) - r_2(t_i)v_1(t_i)}{J_{Sat+Servicing/Debris}^{33}(t_i)},$$

and the resultant change of the quaternions $q_1(t_i)$, $q_2(t_i)$, $q_3(t_i)$, and $q_4(t_i)$ are obtained via the kinematic equations (10). It is worth noting that the effect of impulsive disturbances causes possible jumps of state variables. The state space realization of the satellite in the event of the docking or space debris impacts is therefore a hybrid system which contains both continuous-time and discrete-time components and can be expressed as follows,

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) + G(t)w(t) + \sum_{i=1}^{\infty} E_d(t_i)w_d(t_i)\delta(t - t_i), \qquad \delta x(t_0) = x_0, \qquad (35)$$

where $\delta(t)$ is the Dirac Delta function; the impulsive disturbances $w_d(t_i) \in l_2([t_0, t_f], \mathbb{R}^3)$ the Hilbert space of square-summable $\mathbb{R}^3$-valued sequences with the norm defined by $||w_d||_{l_2}^2 \triangleq \sum_{t_i \in [t_0, t_f]} w_d^T(t_i)w_d(t_i)$; the states $\delta x(t) \in L_{\mathcal{F}_t}^2(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{11}))$ the subset of Hilbert space of $\mathbb{R}^{11}$-valued square-integrable process on $[t_0, t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$ with the norm defined by $||\delta x||_{L_2}^2 \triangleq E\left\{\int_{t_0}^{t_f} \delta x^T(\tau)\delta x(\tau)d\tau\right\} < \infty$; and the control input $\delta u(t) \in L_{\mathcal{F}_t}^2(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^7))$ the subset of Hilbert space of $\mathbb{R}^7$-valued square integrable process on $[t_0, t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$ with the norm defined by $||\delta u||_{L_2}^2 \triangleq E\left\{\int_{t_0}^{t_f} \delta u^T(\tau)\delta u(\tau)d\tau\right\} < \infty$. The impact instants $t_i \in [t_0, t_f]$. Notice that continuous-time bounded coefficients $A(t), B(t)$, and $G(t)$ are defined as before with appropriate dimensions whereas the discrete-time matrix coefficient $E_d(t_i)$ is given by

$$E_d(t_i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (36)$$

It is also interesting to represent the hybrid system (35) in the form of

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) + G(t)w(t), \qquad t \neq t_i, \quad \delta x(t_0) = x_0, \qquad (37)$$

$$\delta x(t^+) = \delta x(t) + E_d(t_i)w_d(t_i), \qquad t = t_i, \quad i = 1, 2, \ldots, \qquad (38)$$

where $\delta x(t_i^+)$ are the values of state dynamics immediately after the impulses, and the time limit t^+ is defined as $\lim_{\epsilon \to 0}(t_i + \epsilon)$ for $\epsilon \in \mathbb{R}^+$. It is important to note that the state variables $\delta x(t)$ are right continuous and

may be left discontinuous due to the discrete time jumps. The continuous and discrete system parts described by (37)-(38) are rewritten as

$$\mathcal{G}_c : \left\{ \ \delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) + G(t)w(t)\,, \quad t \neq t_i\,, \quad \delta x(t_0) = x_0\,, \right. \tag{39}$$

$$\mathcal{G}_d : \left\{ \ \delta x(t^+) = \delta x(t) + E_d(t_i)w_d(t_i)\,, \quad i = 1, 2, \ldots\,, \right. \tag{40}$$

As depicted in Figure 1, a reference orientation is fed into the satellite controller subsystem which then calculates the wheel and thruster commands based on the target orientation and the current orientation. Consequently, it results in a change in reaction wheel angular accelerations and thruster torques and thus the satellite motion is altered. The motion is sensed in the body frame of reference and is changed into an inertial frame by the kinematic equations.
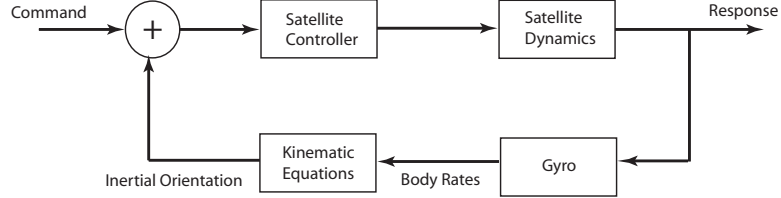

Figure 1. Satellite Functional Block Diagram

IV.   Statistical Control Development

As the satellite is launched into an intended orbit, it never remains in the ideal orientation. In addition to servicing dock and space debris impacts, the external forces present in space created by J2, solar radiation pressure, and aerodynamic torques cause perturbations to this ideal orientation. Thus, the linear time-varying system with finite discrete jumps can now be rewritten

$$\mathcal{G}_c : \left\{ \ dx(t) = (A(t)x(t) + B(t)u(t))\,dt + G(t)dw(t)\,, \quad t \neq t_i\,, \quad x(t_0) = x_0\,, \right. \tag{41}$$

$$\mathcal{G}_d : \left\{ \ x_d(t^+) = x_d(t) + E_d(t_i)w_d(t_i)\,, \quad t = t_i\,, \quad i = 1, 2, \ldots\,, \right. \tag{42}$$

where $A \in \mathcal{C}([t_0, t_f]; R^{n \times n})$, $B \in \mathcal{C}([t_0, t_f]; R^{n \times m})$, $G \in \mathcal{C}([t_0, t_f]; R^{n \times p})$ and the system noise $w(t) \in R^p$ representing all external forces in space other than space debris collisions and servicing docking, is a stationary Wiener process with correlation of increments $E\left\{[w(t) - w(\sigma)][w(t) - w(\sigma)]^T\right\} = W|t - \sigma|$. Also, assume the initial state $x(t_0) = x_0$ is known. For a given (t_0, x_0), associate with the nominal system (41) a finite time integral quadratic form (IQF) random cost $J : \mathcal{C}([t_0, t_f]; R^m) \mapsto R^+$ such that

$$J(u) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} \left[x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\right] d\tau\,, \tag{43}$$

where $Q_f \in R^{n \times n}$, $Q \in \mathcal{C}([t_0, t_f]; R^{n \times n})$ and $R \in \mathcal{C}([t_0, t_f]; R^{m \times m})$ are symmetric and positive semidefinite with $R(t)$ invertible.

As observed by Liberty,[7] all cumulants of any IQF cost in the system state of linear dynamical systems are quadratic affine in the initial state. In the case of state measurement, it is thus fruitful to assume the control input $u(t)$ to be a linear time-varying feedback law given by

$$u(t) = K(t)x(t)\,, \qquad t \in [t_0, t_f] \tag{44}$$

in which $K \in \mathcal{C}([t_0, t_f]; R^{m \times n})$ is an admissible feedback gain whose definition will be clear shortly.

For a given initial condition (t_0, x_0) and an admissible gain K, the kth cost cumulant of random cost J with fixed $k \in Z^+$ is given by Liberty[7] by

$$\kappa_k(t_0, x_0; K) = x_0^T H(t_0, k)x_0 + D(t_0, k)\,, \tag{45}$$

where cumulant-building variables $H(\alpha, k)$ and $D(\alpha, k)$ evaluated at $\alpha = t_0$ are satisfying the differential equations (suppressing time argument of matrix coefficients)

$$\frac{d}{d\alpha} H(\alpha, 1) = -(A + BK)^T H(\alpha, 1) - H(\alpha, 1)(A + BK) - K^T RK - Q \,, \tag{46}$$

$$\frac{d}{d\alpha} H(\alpha, i) = -(A + BK)^T H(\alpha, i) - H(\alpha, i)(A + BK) - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H(\alpha, j) GWG^T H(\alpha, i - j), \tag{47}$$

$$\frac{d}{d\alpha} D(\alpha, i) = -Tr\left\{ H(\alpha, i) GWG^T \right\} \,, \quad 1 \le i \le k \,, \tag{48}$$

with the terminal conditions $H(t_f, 1) = Q_f$, $H(t_f, i) = 0$ for $2 \le i \le k$, and $D(t_f, i) = 0$ for $1 \le i \le k$.

Now it is convenient to denote the right members of the equations (46)-(48) by the symmetric mappings

$$\mathcal{F}_1(\alpha, \mathcal{H}, K) = -(A + BK)^T H(\alpha, 1) - H(\alpha, 1)(A + BK) - K^T RK - Q \,,$$

$$\mathcal{F}_i(\alpha, \mathcal{H}, K) = -(A + BK)^T H(\alpha, i) - H(\alpha, i)(A + BK) - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H(\alpha, j) GWG^T H(\alpha, i - j),$$

$$\mathcal{G}_i(\alpha, \mathcal{H}) = -Tr\left\{ H(\alpha, i) GWG^T \right\} \,, \quad 1 \le i \le k \,,$$

with k-tuple matrix variables

$$\mathcal{H}(\alpha) = (H(\alpha, 1), \ldots, H(\alpha, k)) \,,$$
$$\mathcal{D}(\alpha) = (D(\alpha, 1), \ldots, D(\alpha, k)) \,.$$

Thus, the product system of (46)-(48) is written as follows

$$\frac{d}{d\alpha} \mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}, K) \,, \quad \mathcal{H}(t_f) = \mathcal{H}_f \,,$$

$$\frac{d}{d\alpha} \mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}) \,, \quad \mathcal{D}(t_f) = \mathcal{D}_f \,,$$

where $\mathcal{H}_f = (Q_f, 0, \ldots, 0)$ and $D_f = (0, \ldots, 0)$. Next denote explicitly the dependence of $\mathcal{H}(\alpha)$ and $\mathcal{D}(\alpha)$ on the admissible control gain K by $\mathcal{H}(\alpha, K)$ and $\mathcal{D}(\alpha, K)$. Hence, the performance index in the kCC control problem follows.

Definition 1 *(Performance Index)*
Fix a $k \in Z^+$ and a sequence $\mu = \{\mu_i \ge 0\}_{i=1}^k$ with $\mu_1 > 0$. With (t_0, x_0) given, the performance index of the state feedback kCC control problem is defined by

$$\phi_0 \left(t_0, \mathcal{H}(t_0, K), \mathcal{D}(t_0, K) \right) = \sum_{i=1}^k \mu_i \kappa_i(t_0, x_0; K)$$

$$= x_0^T \sum_{i=1}^k \mu_i \mathcal{H}_i(t_0, K) x_0 + \sum_{i=1}^k \mu_i \mathcal{D}_i(t_0, K) \,, \tag{49}$$

where the real constants μ_i give parametric control freedom.

The class $\mathcal{K}_\mu$ of admissible feedback gains is then defined.

Definition 2 *(Admissible Feedback Gains)*
Let the compact subset $\overline{K} \subset R^{m \times n}$ be the allowable set of gain values. For a given $k \in Z^+$ and a sequence $\mu = \{\mu_i \ge 0\}_{i=1}^k$ with $\mu_1 > 0$, let $\mathcal{K}_\mu$ be the class of $\mathcal{C}([t_0, t_f]; R^{m \times n})$ with values $K(\cdot) \in \overline{K}$ for which there exist solutions to the dynamic equations of motion

$$\frac{d}{d\alpha} \mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)) \,, \quad \mathcal{H}(t_f) = \mathcal{H}_f \,, \tag{50}$$

$$\frac{d}{d\alpha} \mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}(\alpha)) \,, \quad \mathcal{D}(t_f) = \mathcal{D}_f \,. \tag{51}$$

The kCC control optimization problem is now stated.

Definition 3 *(Optimization Problem)*
Suppose both $k \in Z^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then the finite horizon state feedback kCC control optimization may be defined by

$$\min_{K \in \mathcal{K}_\mu} \phi_0\left(t_0, \mathcal{H}(t_0, K), \mathcal{D}(t_0, K)\right), \tag{52}$$

subject to the dynamic equations of motion, for all $\alpha \in T$,

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)), \quad \mathcal{H}(t_f) = \mathcal{H}_f, \tag{53}$$

$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}(\alpha)), \quad \mathcal{D}(t_f) = \mathcal{D}_f. \tag{54}$$

This "Mayer form" optimization problem can be solved by applying an adaptation of the HJB equation and verification theorem of dynamic programming given in Fleming and Rishel.[8] Readers who are interested in seeing the detailed adaptation and associated proofs should refer to the reference.[9]

Theorem 1 *(HJB Equation)*
If there exists an optimal control gain $K^* \in \mathcal{K}_\mu$, then the partial differential equation of dynamic programming

$$\min_{K \in \overline{K}} \left\{ \frac{\partial}{\partial \varepsilon}\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) + \frac{\partial}{\partial vec(\mathcal{Y})}\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})vec(\mathcal{F}(\varepsilon, \mathcal{Y}, K)) + \frac{\partial}{\partial vec(\mathcal{Z})}\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})\,vec(\mathcal{G}(\varepsilon, \mathcal{Y})) \right\} = 0. \tag{55}$$

is satisfied. Here $\mathcal{V}(\cdot, \cdot, \cdot)$ is the value function.

The verification theorem stated in the notation used here is

Theorem 2 *(Verification Theorem)*
Fix a $k \in Z^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Also, let the candidate value function $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ be a continuously differentiable solution of the HJB equation (55) which satisfies the boundary condition

$$\mathcal{W}(t_0, \mathcal{H}_0, \mathcal{D}_0) = \phi_0\left(t_0, \mathcal{H}_0, \mathcal{D}_0\right). \tag{56}$$

Let $K \in \mathcal{K}_\mu$ be any admissible control gain and let $\mathcal{H}(\cdot)$ and $\mathcal{D}(\cdot)$ be the corresponding solutions of the equations of motion (53)-(54). Then $\mathcal{W}(\alpha, \mathcal{H}(\alpha), \mathcal{D}(\alpha))$ is a non-increasing function of α. If $K^* \in \mathcal{K}_\mu$ is a control gain with corresponding solution, $\mathcal{H}^*(\cdot)$ and $\mathcal{D}^*(\cdot)$ of the equations (53)-(54) such that for $\alpha \in [t_0, t_f]$

$$\frac{\partial}{\partial \varepsilon}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) + \frac{\partial}{\partial vec(\mathcal{Y})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha))vec(\mathcal{F}(\alpha, \mathcal{H}^*(\alpha), K^*(\alpha)))$$

$$+ \frac{\partial}{\partial vec(\mathcal{Z})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha))vec(\mathcal{G}(\alpha, \mathcal{H}^*(\alpha))) = 0, \tag{57}$$

then K^* is an optimal control gain in $\mathcal{K}_\mu$ and

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}). \tag{58}$$

where $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is the value function.

For any $\varepsilon \in [t_0, t_f]$, the cumulant-generating states of the product system (53)-(54) defined on the interval $[t_0, \varepsilon]$ have terminal values denoted by $\mathcal{H}(\varepsilon) = \mathcal{Y}$ and $\mathcal{D}(\varepsilon) = \mathcal{Z}$. Observe that the performance index (49) is quadratic affine in terms of arbitrarily fixed x_0, a hypothesis solution to the HJB equation (55) has the form

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = x_0^T \sum_{i=1}^k \mu_i(\mathcal{Y}_i + \mathcal{E}_i(\varepsilon))x_0 + \sum_{i=1}^k \mu_i(\mathcal{Z}_i + \mathcal{T}_i(\varepsilon)),$$

where $\mathcal{E}_i \in \mathcal{C}([t_0, t_f]; R^{n \times n})$ and $\mathcal{T}_i \in \mathcal{C}([t_0, t_f]; R)$ are unknown functions of time. Placing this guess into (55) yields

$$\min_{K \in \overline{K}} \left\{ x_0^T \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)x_0 + \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) + x_0^T \sum_{i=1}^k \mu_i \mathcal{F}_i(\varepsilon, \mathcal{Y}, K)x_0 + \sum_{i=1}^k \mu_i \mathcal{G}_i(\varepsilon, \mathcal{Y}) \right\} = 0. \tag{59}$$

Differentiating the bracket expression in (59) with respect to K yields the necessary condition for an extremum of the performance index (49) on $[t_0, t_f]$,

$$K(\varepsilon, \mathcal{Y}) = -R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r\,, \tag{60}$$

where $\widehat{\mu}_r = \mu_i/\mu_1$ and $\mu_1 > 0$. Applying the feedback gain (60) along the solution trajectories of (53)-(54) yields

$$\frac{d}{d\varepsilon}\mathcal{H}_1(\varepsilon) = -A^T(\varepsilon)\mathcal{H}_1(\varepsilon) - \mathcal{H}_1(\varepsilon)A(\varepsilon) - Q(\varepsilon) + \mathcal{H}_1(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$

$$+ \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_1(\varepsilon) - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\,, \tag{61}$$

$$\frac{d}{d\varepsilon}\mathcal{H}_i(\varepsilon) = -A^T(\varepsilon)\mathcal{H}_i(\varepsilon) - \mathcal{H}_i(\varepsilon)A(\varepsilon) + \mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$

$$+ \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_i(\varepsilon) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\,, \tag{62}$$

$$\frac{d}{d\varepsilon}\mathcal{D}_i(\varepsilon) = -Tr\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\}\,. \tag{63}$$

Replacing (60) into the bracket of (59) yields the minimum

$$x_0^T\left[\sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) + A^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{H}_i(\varepsilon) + \sum_{i=1}^{k}\mu_i\mathcal{H}_i(\varepsilon)A(\varepsilon) + \mu_1 Q(\varepsilon)\right.$$

$$- \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{H}_i(\varepsilon) - \sum_{i=1}^{k}\mu_i\mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$

$$\left. + \mu_1\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) + \sum_{i=2}^{k}\mu_i\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\right]x_0$$

$$+ \sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) + \sum_{i=1}^{k}\mu_i Tr\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\}\,. \tag{64}$$

Having examined (64), $\mathcal{E}_i(\cdot)$ and $\mathcal{T}_i(\cdot)$ are chosen to satisfy the differential equations

$$\frac{d}{d\varepsilon}\mathcal{E}_1(\varepsilon) = A^T(\varepsilon)\mathcal{H}_1(\varepsilon) + \mathcal{H}_1(\varepsilon)A(\varepsilon) + Q(\varepsilon) - \mathcal{H}_1(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$

$$- \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_1(\varepsilon) + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\,, \tag{65}$$

$$\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) = A^T(\varepsilon)\mathcal{H}_i(\varepsilon) + \mathcal{H}_i(\varepsilon)A(\varepsilon) - \mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$

$$- \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_i(\varepsilon) + \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\,, \tag{66}$$

$$\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) = Tr\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\}\,, \qquad 1 \le i \le k\,. \tag{67}$$

At the boundary condition (56), one has

$$x_0^T\sum_{i=1}^{k}\mu_i(\mathcal{H}_{i0} + \mathcal{E}_i(t_0))\,x_0 + \sum_{i=1}^{k}\mu_i(\mathcal{D}_{i0} + \mathcal{T}_i(t_0)) = x_0^T\sum_{i=1}^{k}\mu_i\mathcal{H}_{i0}\,x_0 + \sum_{i=1}^{k}\mu_i\mathcal{D}_{i0}\,.$$

Thus, the initial conditions are found to be $\mathcal{E}_i(t_0) = 0$ and $\mathcal{T}_i(t_0) = 0$ for $1 \leq i \leq k$. Enforcing the initial conditions and comparing (61)-(63) to (65)-(67) uniquely specify

$$\mathcal{E}_i(\varepsilon) = \mathcal{H}_i(t_0) - \mathcal{H}_i(\varepsilon), \qquad \mathcal{T}_i(\varepsilon) = \mathcal{D}_i(t_0) - \mathcal{D}_i(\varepsilon),$$

for any $\varepsilon \in [t_0, t_f]$ and yields the value function

$$\mathcal{W}(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0)) = x_0^T \sum_{i=1}^{k} \mu_i \mathcal{H}_i(t_0) x_0 + \sum_{i=1}^{k} \mu_i \mathcal{D}_i(t_0),$$

which satisfies the sufficient condition (57) of Theorem 2.

Theorem 3 *(Finite Horizon kCC Control Solution)*
Suppose $k \in Z^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^{k}$ with $\mu_1 > 0$ are fixed. Then the optimal state feedback gain K^* that minimizes the performance index $\phi_0(t_0, \mathcal{H}(t_0, K), \mathcal{D}(t_0, K))$ is given by

$$K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \sum_{r=1}^{k} \widehat{\mu}_r H^*(\alpha, r), \qquad \alpha \in [t_0, t_f] \tag{68}$$

where the scalar, real constants $\widehat{\mu}_r = \mu_i/\mu_1$ represent parametric degrees of design freedom and solutions $H(\cdot, r)$ satisfy the backward-in-time matrix differential equations

$$\frac{d}{d\alpha}H^*(\alpha, 1) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T H^*(\alpha, 1) - H^*(\alpha, 1)[A(\alpha) + B(\alpha)K^*(\alpha)]$$
$$- Q(\alpha) - K^{*T}(\alpha)R(\alpha)K^*(\alpha), \tag{69}$$

$$\frac{d}{d\alpha}H^*(\alpha, r) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T H^*(\alpha, r) - H^*(\alpha, r)[A(\alpha) + B(\alpha)K^*(\alpha)]$$

$$- \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} H^*(\alpha, s)G(\alpha)WG^T(\alpha)H^*(\alpha, r-s), \qquad 2 \leq r \leq k \tag{70}$$

with $H^*(t_f, 1) = Q_f$ and $H^*(t_f, r) = 0$ for $2 \leq r \leq k$.

V.   Simulation Results

This section contains several numerical simulations for the statistical controllers described in the previous section. The low earth orbiting satellite is sun-synchronous remote sensing satellite with an inclination of 98.13 degrees, an altitude of 685 km and a total weight of 509 kg. It is assumed that the angular rate and the reaction wheel speeds are available from the rate gyro and the reaction wheel tachometers, respectively. The orbital rate, ω_0 is 0.0010636 rad/s, the total moment of inertia for the spacecraft body is

$$J_C = \begin{bmatrix} 294.62 & 0 & 0 \\ 0 & 129.56 & 0 \\ 0 & 0 & 209.76 \end{bmatrix} kgm^2.$$

This data shows that the x-axis moment of inertia is the largest, followed by that about the z-axis. The moment of inertia about the y-axis is the smallest. Hence for gravity gradient stability, the principal x-axis of the body should be aligned with the orbit normal (pitch), the z-axis should be aligned with the velocity vector, and the y-axis should be the yaw axis. The body axes and the principal axes are assumed to align with each other as the products of inertia are negligible. The moment of inertia of the reaction wheels is

$$J_W = 0.01044 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} kgm^2,$$

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and the parameters for the wheel orientation matrix, L are given as $\alpha = 45$ degrees, $\beta = 54.74$ degrees. Furthermore, the controller design parameters are selected as follows

$$Q = 10 \cdot I_{11 \times 11}, \qquad R = 100 \cdot I_{7 \times 7}, \qquad W = 0.0045 \cdot I_{3 \times 3}.$$

Having been introduced in Section III, the statistical control design equations (69)-(70), of course, are nonlinear, and so represent an interesting study in themselves. In order to assess the trends of performance offered by the use of second and third cost cumulants, it was chosen in the present instance as an approximation approach to the application of the equations (69)-(70). The time-invariant statistical controller has the form

$$K = -R^{-1} B^T \left[H_1 + \mu_2 H_2 + \mu_3 H_3 \right],$$

where H_1, H_2, and H_3 are solutions to coupled algebraic Riccati-type matrix equations, understood as successive approximations to the original equations (69)-(70) for large t_f

$$\begin{aligned}
0 &= A^T H_1 + H_1 A + Q - H_1 B R^{-1} B^T H_1 + \mu_2^2 H_2 B R^{-1} B^T H_2 \\
&\quad + \mu_2 \mu_3 H_2 B R^{-1} B^T H_3 + \mu_2 \mu_3 H_3 B R^{-1} B^T H_2 + \mu_3^2 H_3 B R^{-1} B^T H_3 \\
0 &= A^T H_2 + H_2 A - 2\mu_2 H_2 B R^{-1} B^T H_2 - H_2 B R^{-1} B^T H_1 - H_1 B R^{-1} B^T H_2 \\
&\quad - \mu_3 H_2 B R^{-1} B^T H_3 - \mu_3 H_3 B R^{-1} B^T H_2 + 4 H_1 G W G^T H_1 \\
0 &= A^T H_3 + H_3 A - 2\mu_3 H_3 B R^{-1} B^T H_3 - H_3 B R^{-1} B^T H_1 - H_1 B R^{-1} B^T H_3 \\
&\quad - \mu_2 H_3 B R^{-1} B^T H_2 - \mu_2 H_2 B R^{-1} B^T H_3 + 6 H_1 G W G^T H_2 + 6 H_2 G W G^T H_1.
\end{aligned}$$

The design of a statistical controller using first three cost cumulants was carried out for the attitude hold mode when $\mu_1 = 1.0$, $\mu_2 = 2.0$, and $\mu_3 = 0.15$. Figure 2 illustrates the closed-loop dynamical behavior of the satellite Euler angles, angular rates and wheel speeds with respect to time. From initial values as indicated in Figure 2, the satellite transient responses went to zero within one sixth of the orbital period. Thus, the statistical control method performed satisfactorily at the attitude hold operation. Figure 3 shows the control action from both reaction wheel cluster and thrusters. It was noted that the action of the statistical controller was large at the start and settled down to zero very fast.

In order to evaluate the attitude hold performance of the satellite against space debris impact and servicing dock, some numerical simulations were performed. Figure 4 depicts the effects of two space debris collisions, one with a mass of 0.5 kg and an impact speed of 750 m/s and the other with a mass of 0.3 kg and an impact speed of 800 m/s on the Euler angles, angular rates and wheel speeds of the satellite. The alterations in the satellite pointing angles and angular rates were successfully corrected by statistical control activities via reaction wheels and thrusters as can be seen in Figure 5. And the statistical controller achieved finite-time stabilization.

Finally, a computer-aided software package has been developed to provide a complete statistical description of (43) associated with the Gauss-Markov linear dynamical process (41) in the form of a plot of the probability density function. The software toolbox helps increase the depth of understanding and utility of the statistical control theory in terms of how these statistical controllers affect the overall shape of cost densities after control selection stages are taken place. As illustrated from Figure 8, the shape of the cost density (43) becomes more symmetric as the skewness weighting μ_3 increases to achieve better system performance.

VI. Conclusions

This paper has addressed the application question by applying the statistical control theory to the response control of aerospace structures under stochastic and impulsive disturbances. Although the statistical control method is the new kid on the block, the present results indicate in a most encouraging way that multiple cost cumulants can be effectively utilized to achieve desirable closed-loop system performance. It is hoped that these early results serve as a further stimulus for this very interesting approach.
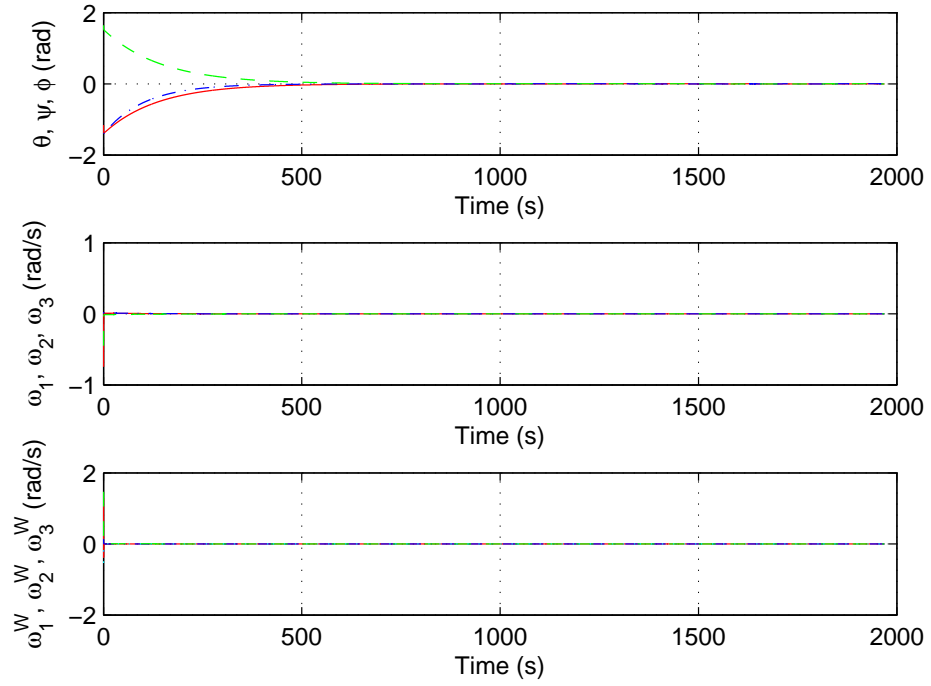
Acknowledgments

Figure 2. Closed-Loop Responses Due To Statistical Controller Design
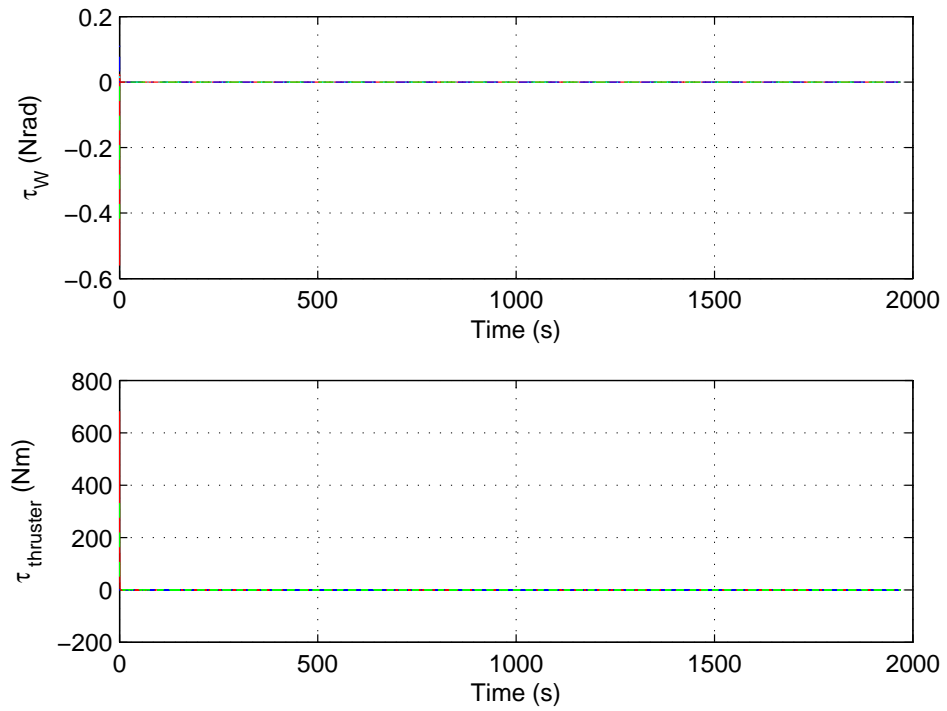
Figure 3. Reaction Wheel and Thruster Torques Due To Statistical Controller Design

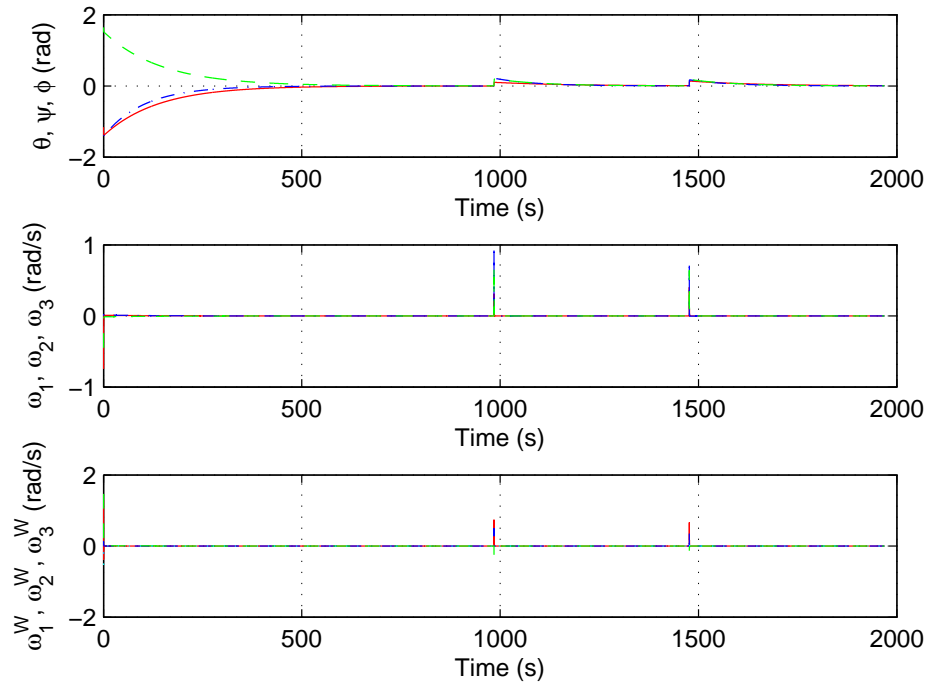American Institute of Aeronautics and Astronautics Paper AIAA-2007-1755

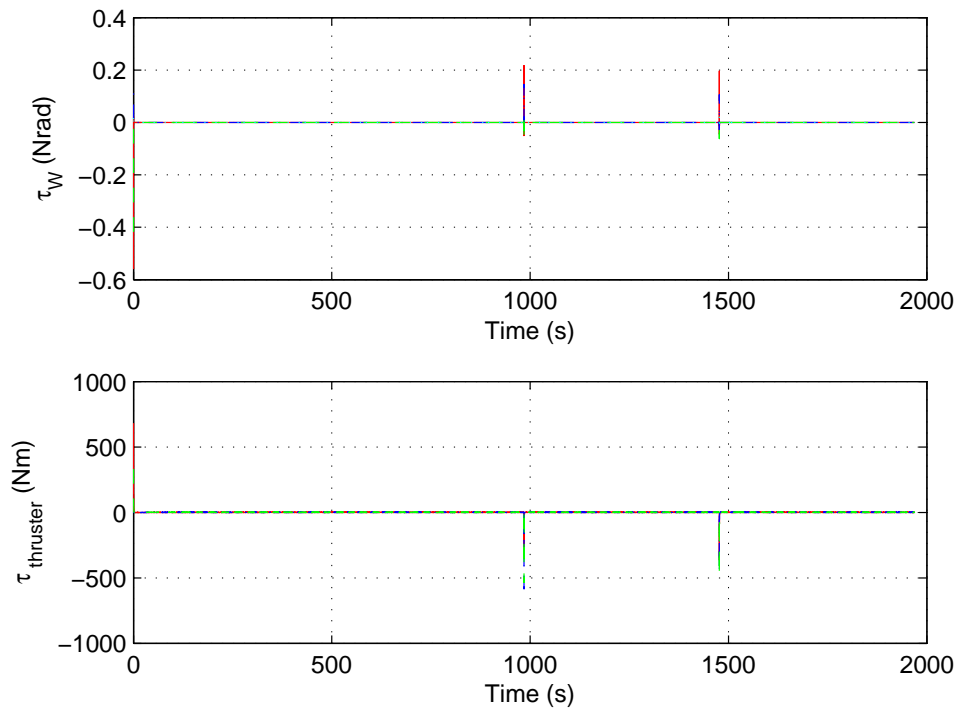**Figure 4. Closed-Loop Responses Due To Space Debris of 0.5 kg and 0.3 kg and Impact Speeds of 750 m/s and 800 m/s**

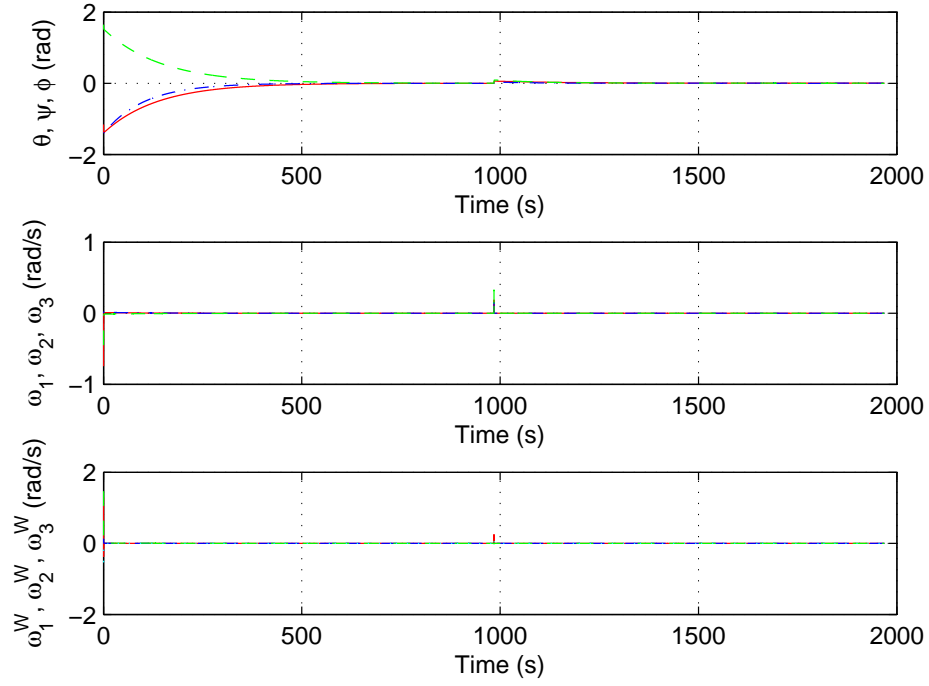**Figure 5. Control Action Due To Space Debris of 0.5 kg and 0.3 kg and Impact Speeds of 750 m/s and 800 m/s**

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Figure 6. Closed-Loop Responses Due To Servicing Satellite of 20 kg and Docking Speed of 5 m/s
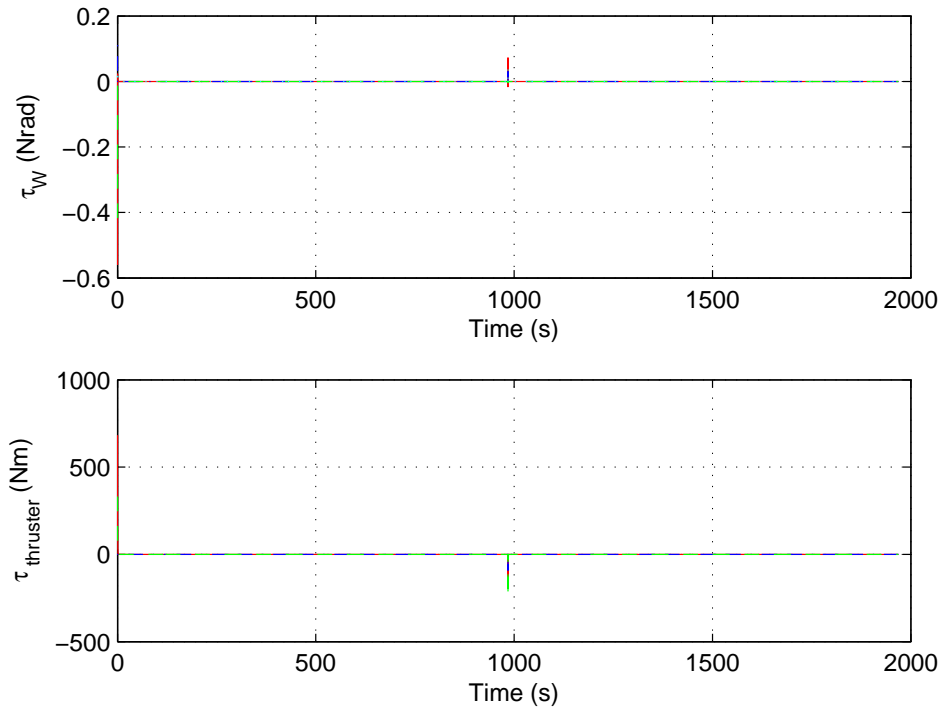
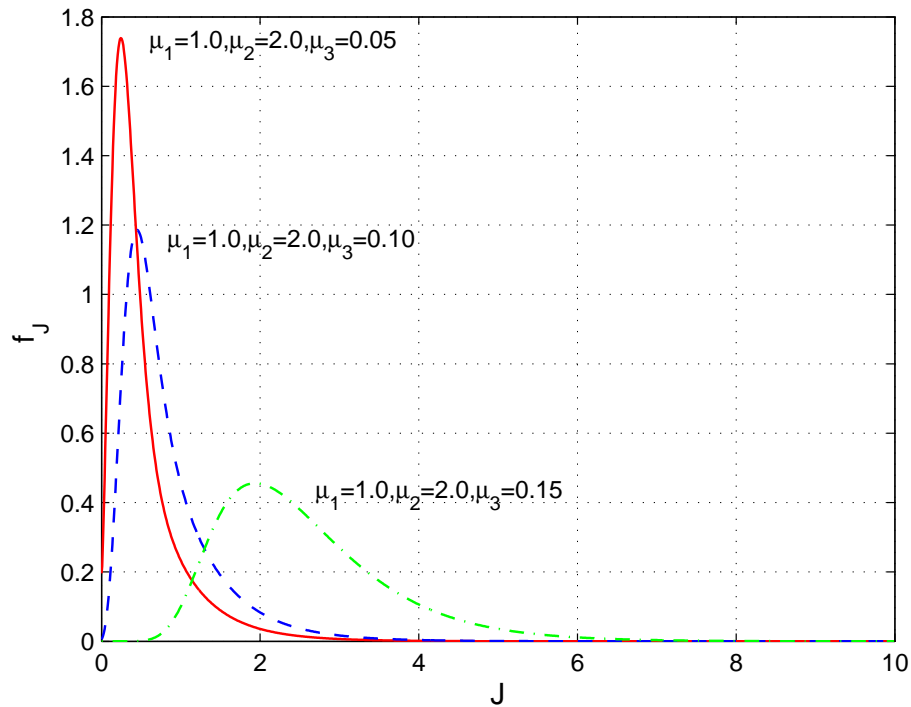**Figure 7. Control Action Due To Servicing Satellite of 20 kg and Docking Speed of 5 m/s**

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Figure 8. Cost Densities Due to Statistical Controllers

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